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Folding processes and solitary waves in structural geology

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Evolution of localized folding patterns in layered elastic and visco-elastic materials is reviewed in the context of compressed geological systems. The thin strut or plate embedded in a visco-elastic medium is used as an archetypal example to describe localized buckles which, in contrast to those from earlier formulations, appear in the absence of triggering imperfections. Structural and material effects are surveyed and important nonlinear characteristics are identified. A brief review of possible methods of analysis is conducted.

1. Introduction

The modelling of geological processes spanning perhaps hundreds of millions of years with well-posed differential equations is fraught with difficulty. There is a considerable gulf between, on the one hand, natural processes undergoing enumerable, large and small, continuous and discontinuous, influences as time progresses and, on the other, easily definable regulatory laws. Nonetheless, when an innovative shift in modelling brings about radical qualitative changes in response, interest is justifiably raised. The recent burst of interest in concepts of regularity and irregularity (fractals and chaos) as applied in the earth sciences (Kruhl 1994) is a good example of this progressive trend.

The process of folding under tectonic compression is an area of structural geology that is perhaps ready for such development. The pioneering work of Biot (1965) models such processes with elastic and viscous struts and plates resting on or within supporting (half-space) media (see figure 5), that themselves are subject to elastic, viscous, or visco-elastic laws. According to the demands and computing power of the time, Biot's modelling was largely restricted to linear differential equations and periodic harmonic deflected forms thus dominate the response. In a move towards physical reality, Mühlhaus (Mühlhaus 1993; Mühlaus *et al.* 1994) later introduced nonlinear terms associated with controlled end displacements rather than loads, but the response apparently remained periodic. This contrasts with observations in the field of more complex phenomena such as spatial localization and quasiperiodicity (Price & Cosgrove 1990), commonly associated with spatial chaos (Champneys & Toland 1993; Hunt *et al.* 1997).

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Such considerations are of real interest to organizations like the CSIRO of Australia, as there is known to be a tendency for mineral deposits to leach into areas of intensified localized folding. This suggests considerable scope for interaction between research workers of wide-ranging experience and expertise, from field geologists with knowledge of the constitutive makeup and nature of folding in real rock systems to numerical analysts at the cutting edge of modern applied mathematics. In such a broad college there is clearly room for contributions of different orders of complexity and rigour.

We attempt here an overview of the process of geological folding, with particular emphasis placed on the emergence of spatial forms that appear as solitary waves when seen at an instant in a slow evolutionary timeframe. We start from a modelling perspective and in the search for localization place the strut formulation of Biot (1965) on or within a *visco-elastic* medium. The elastic component thus allows an instantaneous or near-instantaneous initial response, followed by a slow evolution marking dissipation of stored energy into the viscous part of the support. The simplest mathematical setting for this is a nonlinear partial differential equation (PDE) that is fourth order in space and first order in time. Moreover, if the supporting medium is a half-space it has a non-local constitutive law and the equation is integro-differential.

The paper reviews several possible models for the localized folding of geological structures, from the immediately tractable to the presently intractable, and suggests a number of possible analytical and numerical approaches, from the heuristic to the more strictly rigorous. The emphasis is on localized solutions that occur naturally in 'perfect' systems, rather than those triggered by initial imperfections or perturbations that have received considerable attention, both theoretical and experimental, in the past (see, for example, Cobbold 1975, 1976; Abbasi & Mancktelow 1992; Mancktelow & Abbasi 1992). The scene is set by results from a crude truncation approach, with some asymptotic credentials but extended beyond its range of applicabililty. This foregoes strict rigour for the sake of otherwise unobtainable results and provides convincing qualitative time portraits. In the absence of rigorous single-dimensional methods for half-space formulations, it is introduced as a pointer for more respectable approaches that we hope will follow.

2. Geological evidence

Evidence of folding under in-plane compression appears in a geological context on many scales. Figure 1 shows a specimen of evaporite some 12 cm in length in which periodic folding at different wavelengths is clearly visible. Less obvious but still apparent are decaying fluctuations to the right end of the specimen, which bear a strong resemblance to the fourth-order homoclinic boundary condition described by four complex-conjugate eigenvalues seen elsewhere in this special issue (see, for example, Sandstede, this volume). Localized distortion, either with slowly decaying tails or severe, as in the box folding of figure 2, is thus a known geological feature. Evidence of quasi-periodicity also exists, as in figure 3, which demonstrates a number of interesting structural and material features including further examples of slow amplitude decay and evidence of folding on different wavelengths. Irregularity and self-similarity have recently been recognized as integral to the geological setting (Ord 1994). We note that in all of these cases the behaviour is liable to be significantly influenced by multi-layer (interactive) effects, which we ignore.

A Typical Fold

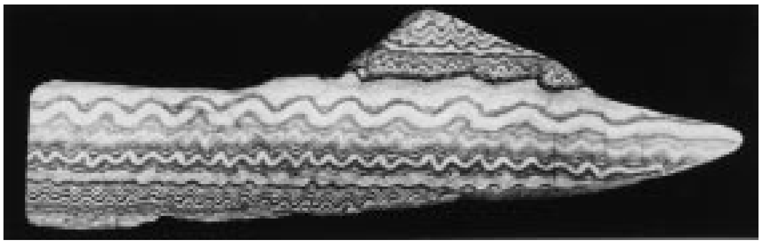

Figure 1. Periodic and decaying folds in a specimen of evaporite (courtesy of J. W. Cosgrove, Imperial College London).

Figure 2. Medium scale box folding seen from the beach at Bude, Cornwall.

Figure 3. Localization and quasi-periodicity in layered Mylor beds at Porthleven, Cornwall.

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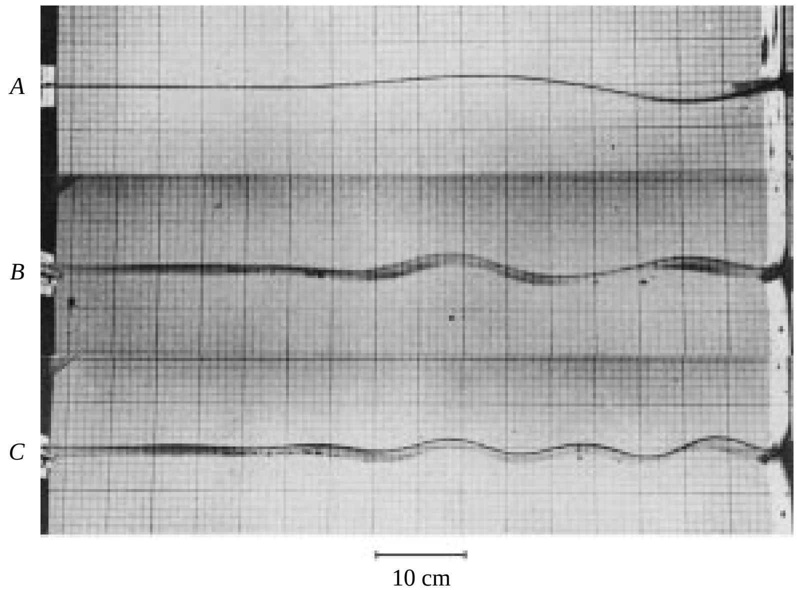

Figure 4. Experiments on an axially compressed thin elastic strip within a viscous medium (after Biot 1965).

3. Modelling considerations

Early attempts to put the folding of geological layers on a firm mathematical footing came from Biot (1965), who treated in some depth the buckling of elastic and viscous layers supported on or within elastic and viscous media. Linear differential equations were predominantly used and, as a result, periodicity tended to govern the responses. Spectral decomposition was central to the approach, leading via the concept of dispersion relations to a focus on the dominant or fastest growing wavelength of buckle. Experiments performed on elastic strips in a viscous oil, shown here in figure 4, are less clearly periodic, however, decaying amplitudes also being apparent. With the evidence also from figures 2 and 3, we can take it that aperiodicity is an accepted feature in the geological setting. While recent work on nonlinear elastic foundations has demonstrated the importance of localized solitary waves (Hunt & Wadee 1991; Champneys & Toland 1993), perfect viscous systems, even under the nonlinear constraint of fixed end displacement, remain periodic for all time (Mühlhaus 1993; Mühlhaus *et al.* 1994); as the compressive load drops to zero the deflected shape apparently evolves through a sequence of longer and longer periodic waves to the final result of a single long wave as $t \rightarrow \infty$.

It has, however, been pointed out by Price & Cosgrove (1990) that neither quantitatively nor qualitatively do the calculations from viscous bedding relations match observed phenomena. Indeed, it is argued that elasticity must be introduced. With this as our cue, we concentrate in what follows on elastic struts on linear and nonlinear visco-elastic foundations.

(*a*) *Structural effects*

Many types of structural form—struts, plates, shells and domes for instance—supported by various media, together with single and multi-layered sandwich structures, blocky structures and other similar archetypes, bear some geological relevance.

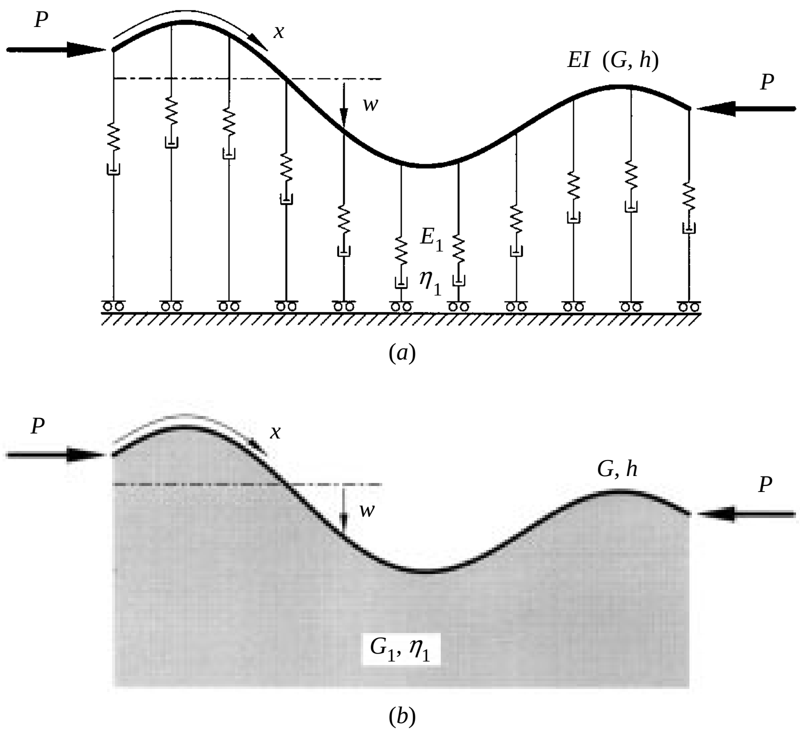

Figure 5. Embedded strut models. (*a*) Winkler foundation; (*b*) half-space bed.

Normal structural assumptions, those associated with thin plate or shell theory, for example, can be called into question. Contact, friction and separation or upheaval (Hunt & Blackmore, this volume) may play major roles. Here we concentrate on a thin strut or plate layer supported on or within elastic, viscous or visco-elastic media, as seen in figure 5. We take a two-dimensional view, with the same form assumed to extend to infinity in each direction in the third.

A strut made from an elastic material of Young's modulus E, of unit width, thickness h and cross-sectional second moment of area I, under a compressive load P and resting on a unspecified bed of vertical resisting force per unit length F, has the governing equation

$$EI[w''''(1-w'^2)^{-1} + 4w'''w''w'(1-w'^2)^{-2} + w''^3(1+3w'^2)(1-w'^2)^{-3}] + Pw''(1-w'^2)^{-3/2} + F = 0, \quad (3.1)$$

where primes denote differentiation with respect to the spatial coordinate x, measured along the length of the strut as seen in figure 5. If deflections are small, and hence nonlinear terms can be ignored, this becomes

$$EIw'''' + Pw'' + F = 0. \quad (3.2)$$

The single differential equation holds only for the disconnected Winkler foundation of figure 5*a*, where resistance to deflection into the bed is strictly local and vertical. For the infinite half-space of figure 5*b*, the situation is significantly different. Resistance now comes from shearing action. A wavy form in the strut causes the bed to shear and is resisted, but there is no resistance to a constant displacement w over infinite length

x. This introduces a wavelength dependence to the bed (Biot 1965) and historically has led to treatment by Fourier analysis.

The modelling implication of such effects is that a fourth-order differential equation in a single spatial variable x is no longer appropriate. The general half-space implies that non-local bedding relations are now expected; the force at a point x along the length depends not just on w at x, but on displacements at points other than x, and integro-differential equations govern. We note that Winkler foundations find limited application for granular, and possibly finite-depth, supporting media, but the half-space is generally considered the more realistic for geological applications. Other attempts to resolve this difficulty include hybrid supporting media like the Pasternak foundation (Kerr 1964).

(i) *Overburden pressure*

If, as in many geological situations, the force F of equations (3.1) and (3.2) contains a significant component of overburden pressure, at first glance it would seem that a layer could buckle under such pressure alone; pressure components top and bottom of the layer would tend to cancel each other, while that at the ends of the layer apparently provides the compressive force necessary for buckling. This, however, conflicts with common sense and is overcome by the following reasoning. Consider a small element dx of a layer of thickness h and unit width, bending through a small angle $d\varphi$, as shown in figure 6. Under constant pressure p, tensile stretching of one face combined with compressive shortening of the other leads to an out-of-balance normal resisting force per unit length,

$$q = \frac{p(r + \frac{1}{2}h)\mathrm{d}\varphi - p(r - \frac{1}{2}h)\mathrm{d}\varphi}{r\mathrm{d}\varphi} = \frac{ph}{r}, \tag{3.3}$$

where r is the radius of bending curvature and h is the thickness of the layer. For the small deflection form of equation (3.2), $w'' = -1/r$, and the net effect is to introduce a term $-phw''$ into the governing differential equation that exactly cancels the perceived compressive force at the ends coming from the overburden pressure. The same conclusion is reached in a continuum context by Mühlhaus *et al.* (1998). The axial load P thus arises entirely from external events like tectonic collision.

(*b*) *Material effects*

The distinction between material and structural behaviour may at times be somewhat blurred, yet it remains a useful exercise to attempt to distinguish between local constitutive laws and structural response.

(i) *Constitutive relations*

The most abundant material phases in the Earth's crust and the upper mantle are quartz/feldspar and olivine, respectively. They deform permanently by a number of alternative, often competing, mechanisms, the most important in a geological/geophysical context being: (1) diffusional flow (Nabarro-Herring creep, Coble creep); (2) power-law creep by dislocation glide; (3) collapse at yield strength. Each can be described by a rate equation relating the strain rate $\dot{\gamma}$ to the stress τ, the temperature T and the structure of the material at that instant (see, for example, Frost & Ashby 1982; Estrin 1996), where dots denote differentiation with respect to time. In three-dimensional applications, the rate equations have to be complemented by a flow rule (in the simplest cases, the Prandtl–Reuss flow rule, see Estrin 1996)

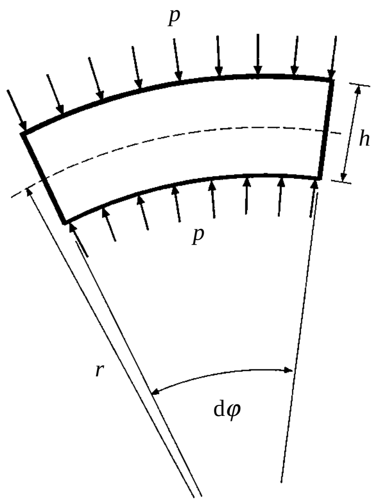

Figure 6. Elemental system.

and $\dot{\gamma}$ and τ are usually defined as

$$\dot{\gamma} = (2D_{ij}D_{ij})^{1/2} \quad \text{and} \quad \tau = (\tfrac{1}{2}\sigma'_{ij}\sigma'_{ij})^{1/2}, \tag{3.4}$$

where D_{ij} is the stretching and σ'_{ij} is the deviatoric part of Cauchy's stress tensor σ_{ij}. A rate law which captures all of the above mechanisms reads

$$\dot{\gamma} = \frac{a}{kT}\left(\frac{\tau}{\tau_0}\right)^n \exp\left(-\frac{Q - pV}{kT}\right). \tag{3.5}$$

Here τ_0 is the yield strength at zero Kelvin, $p = -\frac{1}{3}\sigma_{kk}$ is the pressure, Q is the activation energy at $p = 0$, V is the activation volume for diffusion, k is the Boltzmann constant and a is a parameter with the dimension of power which, in general, is pressure dependent, although this dependency only becomes significant for pressures larger than one tenth of the elastic bulk modulus. Pre-exponential temperature dependency is also mostly negligible and hence a/kT can be replaced in (3.5) by a constant reference strain rate $\dot{\gamma}_0$. The reference stress τ_0 may also depend on the activities of water and of oxygen, $a_{\mathrm{H_2O}}$ and $a_{\mathrm{O_2}}$, respectively (Hobbs 1981; Ord & Hobbs 1989), but this dependency is very weak and is probably only important in zones of high gradients of the activities. If molar values for Q and V are to be used then k must be replaced by the gas constant $\mathrm{R} = 8.314\ \mathrm{J\ mol^{-1}\ K^{-1}}$. Typical orders of magnitude for Q and V are 150–550 kJ mol^{-1} and $7\text{–}30 \times 10^{-6}\ \mathrm{m^3\ mol^{-1}}$, respectively. Lithostatic pressures approach Q/V in magnitude only for depths greater than 700–800 km. Hence, under crustal and upper mantle conditions (roughly taken as depths of up to 300 km), the pressure dependency of the exponent in (3.5) can safely be neglected. The power law exponent n is equal to unity for diffusional flow (Nabarro–Herring creep, Coble creep) and larger than unity for power-law creep, typical values for crustal and mantle material ranging between $2 < n < 3$.

If more detailed analysis of the deformation of geological structures is required, the influence of elastic deformations in the rate law must be considered. Here we are concerned with the evolution of buckling folds, which usually take place on spatio-temporal scales within which the temperature can be regarded as constant. Neglecting the pressure sensitivity as discussed above, the relevant rate law for constant

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temperature reads

$$\dot{\gamma} = \frac{\dot{\tau}}{\mu} + \frac{\tau}{\eta}, \tag{3.6}$$

where μ is the elastic shear modulus, the viscosity η is given by

$$\eta = \frac{\tau_0}{\dot{\gamma}_0}\left(\frac{\tau}{\tau_0}\right)^{1-n}. \tag{3.7}$$

For simplicity we assume in the following that $n = 1$ and

$$\tau = \hat{\tau}\mathrm{e}^{\omega t} \quad \text{and} \quad \gamma = \hat{\gamma}\mathrm{e}^{\omega t}. \tag{3.8}$$

Inserting into (3.6) and re-arranging yields the relation $\hat{\tau} = \hat{\mu}\hat{\gamma}$, where

$$\hat{\mu} = \frac{\omega\eta}{1 + \omega\eta/\mu}. \tag{3.9}$$

The material is predominantly elastic if $\omega\eta >> \mu$ and predominantly viscous if $\omega\eta << \mu$.

Biot (1965, p. 419) investigates the folding of an elastic plate in a viscous medium. The growth coefficient ω_{d} of the fastest growing periodic wave is obtained as

$$\omega_{\mathrm{d}} = \frac{\sigma}{6\eta_1}\sqrt{\frac{\sigma}{\mu}}, \tag{3.10}$$

where η_1 is the viscosity of the embedding medium, μ is the shear modulus of the (incompressible) plate material and the axial stress $\sigma = P/h$ is the driving force of the instability. Inserting (3.10) into $\omega\eta/\mu$ gives the criterion

$$\frac{\eta}{6\eta_1}\left(\frac{\sigma}{\mu}\right)^{3/2} >> 1, \tag{3.11}$$

for predominantly elastic behaviour of the plate during folding.

The single layer folding model considered here comprises a thin, axially pre-stressed layer embedded in a visco-elastic medium. The early stages of the fold amplitude evolution can be understood as a small perturbation of the homogeneously stressed ground state. The tangent shear modulus is obtained in symbolic form as

$$\hat{\mu} = \frac{\mu\eta\dfrac{\partial}{\partial t}}{\mu + \eta\dfrac{\partial}{\partial t}}, \tag{3.12}$$

where viscosity η is given by (3.7) and $\tau = \sigma/(2\sqrt{2})$ for uniaxial plane strain, σ again being the magnitude of the axial pre-stress. The instantaneous bending stiffness of the strut in equation (3.1) is then

$$EI = \tfrac{1}{3}\hat{\mu}h^3. \tag{3.13}$$

The following section gives an interpretation of such basic rheologies in terms of structural elements.

(ii) *Simplified rheologies*

Standard rheological models are available for application in the geological context, three of the most common being shown in figure 7. The Maxwell fluid of figure 7*a*,

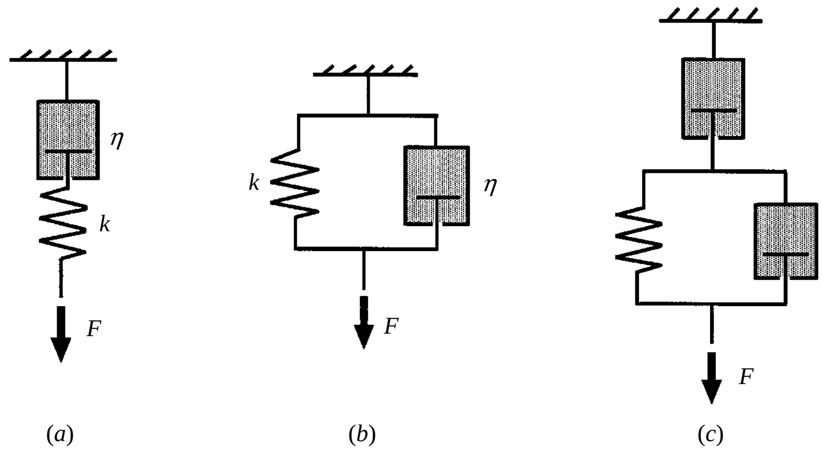

Figure 7. Rheological models and associated responses. (*a*) Maxwell fluid; (*b*) Kelvin–Voigt solid; (*c*) combined fluid.

comprising a spring and dashpot in series, is the most immediately relevant for our purposes. On application of dead load F it shows an instantaneous elastic displacement in the spring, followed by a constant rate of displacement in the dashpot. On application of a constant displacement the spring again responds instantly, but over time gradually unloads into the dashpot such that F falls off exponentially. The governing differential equation is

$$\dot{w} = \frac{1}{k}\dot{F} + \frac{1}{\eta}F, \tag{3.14}$$

where k is the spring stiffness and η is the dashpot viscosity (cf. equation (3.6)).

In contrast, the Kelvin–Voigt solid of figure 7*b*, comprising spring and dashpot in parallel, has the ability to store strain energy. It is represented by the differential equation

$$F = kw + \eta\dot{w}. \tag{3.15}$$

The combined fluid of figure 7*c* slows down the instantaneous elastic response of the Maxwell unit, at the expense of a governing differential equation of second order with the general form

$$F + p_1\dot{F} = q_1\dot{w} + q_2\ddot{w}, \tag{3.16}$$

where p_i and q_i are material constants relating to spring stiffnesses and dashpot viscosities. These are defined more completely in Roscoe (1950), who shows that a general arrangement of such linear springs and dashpots can be reduced to one of two possible canonical forms.

(iii) *Nonlinear models*

Nonlinearities of many kinds can be introduced to the single dimensional rheological models of figure 7. Figure 8 shows three typical types produced by simple geometric effects, all of which can occur naturally. The hardening response of figure 8*a* might be associated, for example, with closing of voids in porous medium, while the softening behaviour of figure 8*b* is similar to that of the shallow arch (Thompson & Hunt 1984). The bilinearity of figure 8*c* is produced by an elastic bifurcation, but

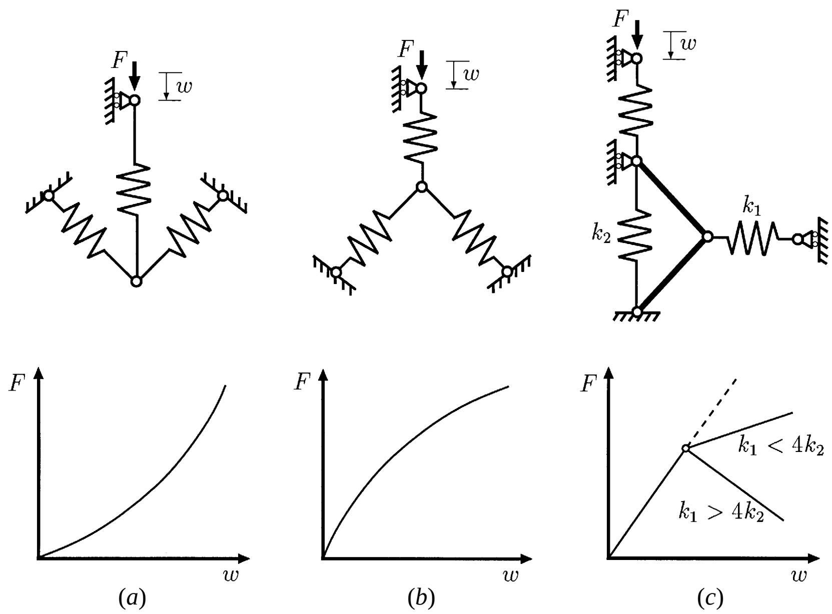

Figure 8. Nonlinear elastic models and associated responses: (*a*) hardening; (*b*) softening; (*c*) bilinear.

in the absence of unloading can be used to mimic elastoplasticity or quasi-brittle fracture (Hunt & Baker 1995).

For purely elastic systems, nonlinearities can be relatively straightforward to introduce and interpret, but when elastic and viscous parts are present it is necessary to identify just how the nonlinearity is to enter the modelling process. In an attempt to reproduce the softening of figure 8*b* in a Winkler foundation, we might, for example, introduce $F = w_\mathrm{e} - w_\mathrm{e}^3$ into the elastic part of the Maxwell element of figure 7*a*, w_e being the displacement in the spring alone; this can be taken either as a simple cubic law in its own right, or as the leading nonlinear term of a more general power expansion. The total deflection w is difficult to determine from this perspective, but is readily obtained if the relation is reversed to give $w_\mathrm{e} = F + F^3 + \cdots$. This more convenient expansion may, however, have a reduced range of validity; unlike its counterpart it cannot, for instance, represent a response that continues over a maximum into a regime of negative stiffness (Whiting 1996).

We note in passing that a nonlinearity on the viscous part, or the whole of w as in Hunt *et al.* (1996*a*), would have a growing effect over time. The viscous part would continue to deflect and involve the nonlinearity to an ever-increasing extent.

(iv) *Half-space formulations*

Linear and nonlinear rheological models are readily adapted to Winkler foundations but not so easily to a half-space. Biot (1937) took a Fourier expansion for a general load on the boundary of an elastic half-space,

$$F(x) = \sum_{0}^{\infty} q_\mathrm{m} \cos k_\mathrm{m} x, \qquad (3.17)$$

and passed it through the linear elastic plane strain equation ($\nabla^4 \Phi = 0$, where Φ is a stress function) to obtain a relation between q_m and corresponding amplitude of displacement $a_{\rm m}$ given by

$$w(x) = \sum_{0}^{\infty} a_{\rm m} \cos k_{\rm m} x, \tag{3.18}$$

as follows

$$q_{\rm m} = -4\mu_1 k_{\rm m} a_{\rm m}, \tag{3.19}$$

where μ_1 is the shear modulus of the foundation material. The appearance of wavenumber $k_{\rm m}$ marks the fact that the resisting force is wavelength dependent, a feature that is linked to the non-local nature of the bedding support; this contrasts directly with the Winkler formulation in which $k_{\rm m}$ would be absent. A similar exercise for a purely viscous foundation of viscosity η_1 leads to

$$q_{\rm m} = -4\eta_1 k_{\rm m} \dot{a}_{\rm m} \tag{3.20}$$

and, extending the characteristic of equation (3.6) to the linear Maxwell visco-elastic half-space (Hunt *et al.* 1996*a*),

$$\dot{q}_{\rm m} + r q_{\rm m} = -4\mu_1 k_{\rm m} \dot{a}_{\rm m}, \tag{3.21}$$

where $r = \mu_1/\eta_1$.

(v) *Elastica nonlinearities*

Mühlhaus (1993) suggests that the natural nonlinearities of large-deflection bending seen in equation (3.1) generate the following equation in tangent angle φ:

$$\varphi'''' + 2\varphi'' + \tfrac{3}{2}\varphi'^2\varphi'' + F = 0, \tag{3.22}$$

where Fourier decomposition would, in principle, allow the displacement w in the Maxwell relation (3.14) to be replaced by φ, giving the general non-dimensional form

$$\dot{F} + F = \dot{\varphi}. \tag{3.23}$$

Numerical studies of the elastica on Winkler foundation with a Maxwell law, useful for comparison purposes, are given by Whiting & Hunt (1997).

(*c*) *Single governing equation*

For Winkler media, equation (3.1), or its linearized equivalent (3.2) about $w = F = 0$, can be combined simply with constitutive relations such as (3.14) to provide a single governing PDE in one spatial and one time dimension. For example, if the compressive load P remains constant, after a little symbolic manipulation of (3.2) and (3.14), the single equation

$$EI\dot{w}'''' + P\dot{w}'' + k\dot{w} + \frac{k}{\eta}(EIw'''' + Pw'') = 0 \tag{3.24}$$

appears (Hunt *et al.* 1996*a*). Half-space formulations are not so straightforward, however, and lead, in general, to partial integro-differential equations; these are then open to analysis either by Fourier expansion as described above or by general application of Green's theorem. Such approaches work well for linearized systems (Biot 1965) and possibly also those with slowly varying amplitudes and phase. However, they are liable to be called into question when thoroughly localized responses are involved.

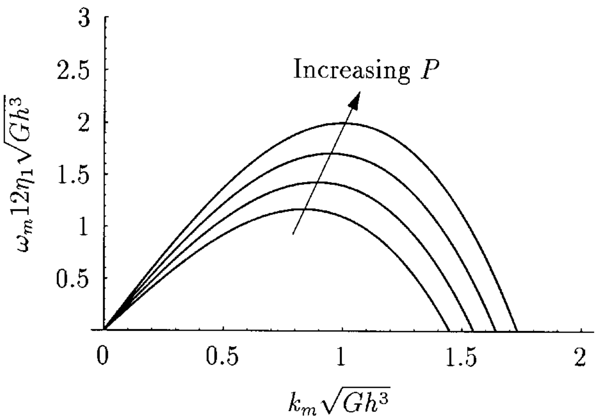

Figure 9. Dispersion relation for a strut on a viscous half-space.

4. Methods of analysis

Analytical and numerical methods for solving such differential and integro-differential equations are now briefly reviewed. Linear and nonlinear systems often demand quite different analytical approaches.

(*a*) *Linearized systems*

Biot's (1937) approach, expressed in Fourier transforms using equation (3.20) for an elastic layer of shear modulus μ and thickness h compressed by a load P and supported by a viscous half-space of viscosity η_1, leads to the dispersion relation of figure 9, where growth rate ω_{m} is given by the eigenmode, $a_{\mathrm{m}} = A_{\mathrm{m}}\mathrm{e}^{\omega_{\mathrm{m}}t}$. A finite range of wavelengths is involved, from the infinitely long ($k_{\mathrm{m}} = 0$), through the *dominant wavelength* of equation (3.10), for which ω_{m} is a maximum,

$$\omega_{\mathrm{d}} = \frac{P}{6\eta_1}k_{\mathrm{d}}, \quad \text{where} \quad k_{\mathrm{d}} = \frac{1}{h}\sqrt{\frac{P}{\mu h}} \tag{4.1}$$

(Biot 1965), and ending at a P-dependent short wavelength cut-off value. Plotted in non-dimensional form these curves superimpose on one another (Hunt *et al.* 1996*a*) and so it is possible to remove P from all consideration.

This contrasts with the (non-dimensionalized) dispersion relations for a visco-elastic medium of viscosity η_1 and shear modulus μ_1 shown in figure 10. Here the minimum critical buckling load $P^{\mathrm{C}}_{\mathrm{min}}$ and its corresponding mode k_{min} are given by

$$P^{\mathrm{C}}_{\mathrm{min}} = \frac{6}{k_{\mathrm{min}}}\mu_1, \quad \text{where} \quad k_{\mathrm{min}} = \frac{1}{h}\sqrt[3]{\frac{6\mu_1}{\mu}}, \tag{4.2}$$

$p = P/P^{\mathrm{C}}_{\mathrm{min}}$, $\tilde{k}_{\mathrm{m}} = k_{\mathrm{m}}/k_{\mathrm{min}}$, $\tilde{t} = rt$, $\tilde{r} = r/\omega_{\mathrm{d}}$ and $\tilde{\omega}_{\mathrm{m}}$ comes from the eigensolution $a_{\mathrm{m}} = A_{\mathrm{m}}\mathrm{e}^{\tilde{\omega}_{\mathrm{m}}\tilde{t}}$, as before. As p approaches the critical elastic buckling value $p^{\mathrm{C}} = 1$, the rate of growth of the dominant mode amplifies in a kind of resonance phenomenon, until at the critical load it grows infinitely fast and expresses the instantaneous elastic response in the critical buckling mode.

(*b*) *Nonlinear systems*

Dispersion relations are by their nature linear concepts, being based on comparisons between simple spectral components of the displacement pattern; interactions

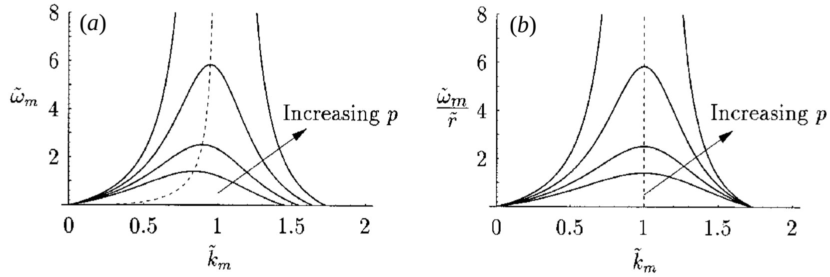

Figure 10. Dispersion relation for a strut on a visco-elastic half-space.

between components are never considered. Linear equations are never able to describe the evolution of localized forms that is of primary concern here: when such nonlinear features are involved, quite different analytical and numerical tools are required (see, for example, Beyn 1990; Champneys & Toland 1993). The added complexity of wavelength dependence, associated with the concept of a half-space, is again based on spectral decomposition and linear in concept. Nonlinear analysis can, however, sometimes draw on such a view by extending it in the sequential asymptotic sense, as, for example, in the use of harmonic fluctuations with slowly varying amplitude, phase and indeed wavelength. Provided the nonlinearities are polynomial and of moderate order, Fourier transformation methods are known to work well for viscous media (Mühlhaus *et al.* 1998), but are less useful for the severe localization associated with brittle (fracturing) structures.

(i) *Trial functions*

Trial functions, as seen, for example, in Galerkin or Rayleigh–Ritz formulations, might prove useful in overcoming problems of wavelength dependence; wavelengths are built in to the modelling process and have the potential to vary (see, for example, Wadee *et al.* 1997). However, Galerkin procedures involving wavelength variation are yet to be successfully devised for such problems, whereas Rayleigh–Ritz procedures depend on conservation of energy which is contravened for visco-elastic foundations. Although it is felt that useful analytical approaches could develop from such procedures, at present, under the banner of trial function methods, there only exists, as far as we are aware, a crude truncation approach (Hunt *et al.* 1996*a*). Results from this process should be treated with caution, taken over finite time perhaps merely as qualitative pointers towards possible behaviour; it does, however, provide reasonably convincing time portraits.

The process is described fully in Hunt *et al.* (1996*a*) and summarized in Hunt *et al.* (1996*b*). Under the constraint of constant end-shortening, trial functions are fed into a governing nonlinear PDE with a zero right-hand side, and enough coefficients of the resulting set of functions put to zero to match a number, typically seven, unknown variables. A simple marching algorithm then gives the evolution over time. Figure 11 shows a typical output based on the PDE

$$\dot{w}'''' + 3p\dot{w}'' + \dot{\overline{[2\beta(w - w^3)]}} + (w'''' + 3pw'') = 0. \quad (4.3)$$

Wavelength dependence is represented by the explicit appearance of β, as defined by the trial function

$$w = A \operatorname{sech} \alpha x \cos \beta x + B \operatorname{sech} \alpha x \tanh \alpha x \sin \beta x + Cx \operatorname{sech} \alpha x \sin \beta x + Dx \operatorname{sech} \alpha x \tanh \alpha x \cos \beta x. \quad (4.4)$$

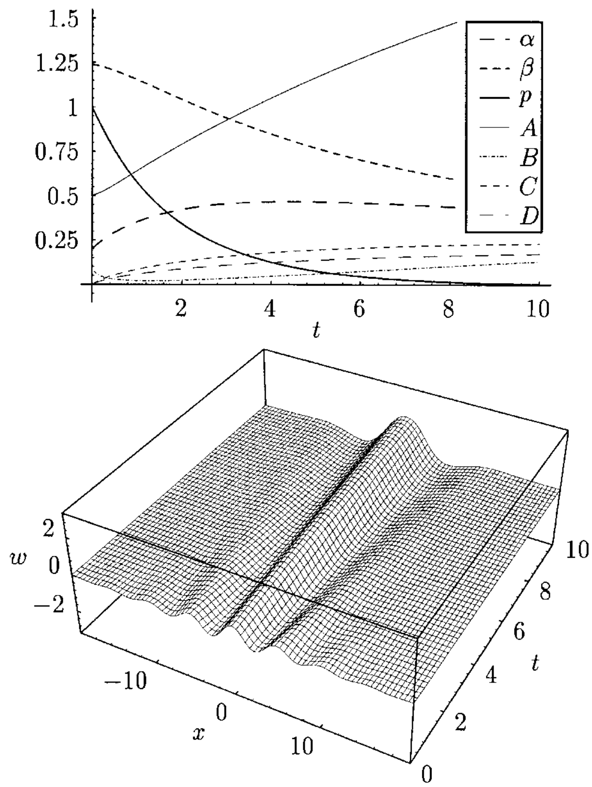

Figure 11. Seven-variable evolution of a localized shape under constant end displacement. Based on the elastic strut of equation (3.2) on a visco-elastic half-space with $EI = 1$, $P = 3p$ and $F = 2\beta(w - w^3)$. (See Hunt *et al.* 1996*a*.)

The free variables are A, B, C, D, α, β and p. Six first-order ODEs in time arise from setting the coefficients of $\operatorname{sech}\alpha x\cos\beta x$, $\operatorname{sech}\alpha x\tanh\alpha x\sin\beta x$, $\operatorname{sech}^3\alpha x\cos\beta x$, $\operatorname{sech}^3\alpha x\tanh\alpha x\sin\beta x$, $x\operatorname{sech}\alpha x\sin\beta x$ and $x\operatorname{sech}\alpha x\tanh\alpha x\cos\beta x$ to zero, with the seventh coming from the constraint of constant end-shortening. As the load p drops to zero, the localized wave pattern opens out to release the bending strain energy of the elastic strut into the foundation. The eventual outcome, not portrayed here, is a single long wave which, if length is taken to infinity, has zero amplitude.

(ii) *Numerical methods*

In the modern context, numerical methods provide the most obvious way of tackling the nonlinear governing equations that arise in modelling of geological systems. Yet there is little published work of which we are aware that relates directly to the solitary waveforms of interest here. Typically, although nonlinearities and localization are now routinely considered, the latter is usually taken in the sense of a thorough localization, as in brittle or quasi-brittle fracture (Bažant & Cedolin 1991; Leroy & Triantafillydis 1996). There remain a number of fundamental problems to be addressed. It is generally accepted in geological circles that half-space formulations are the most credible, but these carry the penalty of non-local regulatory laws leading in the first instance to integro-differential equations. In the linear context such equations can be reduced to wavelength dependent PDEs as discussed above (Biot 1965), but for station-based approaches such as the collocation method this is numerically awkward; the equation to be solved depends on the solution. Green's function methods suggest one possible way forward, yet these too are limited by being linear in concept.

The expansive alternative is to introduce a second spatial dimension as in a finite element (FE) formulation, but with time providing a third dimension this is likely to be expensive in computing power. Such formulations are popular (Lewis & Williams 1978; Williams *et al.* 1978) but may be problematical (Cobbold 1977). They inevitably suffer from the fact that a plate or a strut is long and thin, and must be subdivided into many elements to achieve the necessary accuracy for short wavelength deformation. The supporting medium, where deformation is minimal, would then require subdivision into something like n^2 (or n^3) elements, n being the number of beam elements. A combination of FE and boundary elements might prove more efficient, but boundary elements only work well if the nonlinearity in the supporting medium is unimportant. FE formulation might also require remeshing at larger amplitudes, because of mesh distortion. One recent relevant thesis is due to Breekman (1994), who models lithospheric and crustal elasto-viscoplastic deformation using finite elements.

We note finally that it is certainly possible that the problem is suited to analysis by wavelets, but this approach is yet to be explored. A second promising alternative is a particle-based, as opposed to mesh-based, approach to discretization (Mühlhaus & Hornby 1996).

5. Concluding remarks

The paper emphasizes that nonlinear PDEs and integro-differential equations lie at the heart of the problem of geological folding. The field is in its infancy: Winkler foundations can be formulated, but the more relevant half-space has not apparently been tackled rigorously in the nonlinear range. Where material softening is involved, it is demonstrated that spatial localization is likely to be central to the response over time. Trial function methods supply perhaps the best analytical way forward; the relatively short length scales and long time scales involved suggest that space and time may be able to be handled separately. Likely also to be significant to real geological systems are constitutive laws with hardening responses, or those that initially soften followed by hardening. Quasi-periodicity, seen in the field but not part of the present description, would then naturally enter the frame; such matters are, however, out of the remit of this special issue.

The review generally raises more questions than it answers. It is worthy of note that a recent UK EPSRC grant under the Applied Nonlinear Mathematics Initiative, to address some of the matters raised, commenced in January 1997.

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